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Parabolic Potential Theory

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In this paper we study solutions of heat equations, stressing the potential theoretic point of view. In particular we are interested in properties which have well-known Laplacian counterparts. Many have worked in this direction; however because of the non-self-adjointness of the heat equation and the time lag, the problems seem difficult.

We obtain some surprising counterexamples: there exists a heat potential $v \not\equiv +\infty$ in $\{(x, t)\colon\ t > 0\}$ such that for every x between 0 and 1, $\limsup_{t \to 0^+} v(x, t) = +\infty$; there exists a positive solution h of the heat equation in $\{(x, t)\colon x > 0\}$ given by a continuous measure μ on $x = 0$, such that $\liminf_{x \to 0^+} h(x, t) = 0$ for every real number t; the graph of a decreasing function $t = f(x)$, $0 < x < 1$, can have zero heat capacity.

Among the positive results are a ratio Fatou theorem, a result on sets of heat capacity zero, and comparison of sets of parabolic measure zero and length zero. We also indicate some questions left open.

0. PRELIMINARIES

In this paper, we denote by R and D the regions $\{(x, t)\colon\ x > 0, -\infty < t < \infty\}$ and $\{(x, t)\colon t > 0 \text{ and } -\infty < x < \infty\}$, respectively. We say u is a parabolic function in a domain Ω, if $(\partial^2/\partial x^2 - \partial/\partial t)\, u = 0$ in Ω. Let W be the fundamental solution of the heat equation, defined by

$$\begin{aligned} W(x, t; y, s) &= [4\pi(t-s)]^{-1/2} \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] && \text{for} \quad t > s \\ &= 0, && \text{for} \quad t \leqslant s, \end{aligned} \tag{0.1}$$

which is a solution of $\partial^2/\partial x^2 - \partial/\partial t = 0$ in $E^2 \backslash (y, s)$ for each fixed (y, s). Let K be the heat kernel in R with pole at $(0, s)$, that is,

$$\begin{aligned} K(x, t; 0, s) &= (4\pi)^{-1/2} x (t-s)^{-3/2} \exp\left[-\frac{x^2}{4(t-s)}\right] && \text{for} \quad t > s \\ &= 0 && \text{for} \quad t \leqslant s. \end{aligned} \tag{0.2}$$

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The following theorem is probably well known, however, we cannot find it in the literature. We give a proof in the Appendix for completeness.

THEOREM A. *If u and v are positive parabolic functions in R and D, respectively, then there exist a number $p \geqslant 0$, Borel measures dU and dV on ∂R and ∂D, respectively, Borel measure $d\mu$ on $0 < \lambda < \infty$, so that u and v have the integral representations*

$$u(x, t) = \int_{\partial R} K(x, t; 0, s)\, dU(0, s) + \int_0^{\infty} \sinh \lambda x e^{\lambda^2 t}\, d\mu(\lambda) + px \qquad (0.3)$$

where the second integral vanishes continuously on ∂R, and

$$v(x, t) = \int_{\partial D} W(x, t; y, 0)\, dV(y, 0). \qquad (0.4)$$

Moreover, the number p, the measures dU, $d\mu$ and dV in (0.3) *and* (0.4) *are unique.*

For the maximum principle on solutions of heat equation or adjoint heat equation in a rectangular region, we refer the reader to [4, p. 152].

1. FATOU-TYPE THEOREMS

Corresponding to nontangential limit for harmonic functions, we discuss parabolic limit for parabolic functions. We say a function v in D has parabolic limit L at $(y, 0)$ if for each $a > 0$, $\lim v(x, t) = L$ as $(x, t) \to (y, 0)$ inside $(t > a(x - y)^2)$. We say a function u in R has parabolic limit L at $(0, s)$ if for each $a > 0$, $\lim u(x, t) = L$ as $(x, t) \to (0, s)$ inside $(|t - s| < ax^2)$. The following Fatou theorem is a simple consequence of Theorem A and a theorem of Kemper [6, p. 259].

THEOREM B. *Let u be a nonnegative parabolic function in R (or D), given by a measure U on ∂R (or ∂D). Then u has finite parabolic limit at Lebesgue almost every point on ∂R (or ∂D) which equals the derivative dU/dt (or dU/dx) whenever the latter exists and is finite.*

Using a theorem of Koranyi and Taylor on relation between fine limit and admissible limit, we can also give a proof of Theorem B via a fine topology approach; see [7].

In [1, Theorem 5.1], Doob proved the following probabilistic version of the Fatou theorem:

THEOREM C. *If u is a nonnegative parabolic function in a finite open set Ω, then u has a finite limit along almost every Brownian trajectory with decreasing t, from any point of Ω to $\partial\Omega$.*

In [3, Theorem 3.1], Doob proves the following relative Fatou theorem for functions in D.

THEOREM D. *Suppose v and h are two positive parabolic functions in D corresponding to two measures V and H on ∂D. Then v/h has parabolic limit equal to the Radon–Nikodym derivative dV/dH at H-almost every point on ∂D.*

For positive parabolic functions in R, the relative Fatou theorem that one expects is not true, as one can see easily from the example: $u(x, t) \equiv 1$, $h(x, t) = K(x, t; 0, 0)$, H is the unit point measure at $(0, 0)$ and $\lim_{x\to 0^+} (u(x, 0)/h(x, 0)) = +\infty$. The real reason behind this example is $\liminf_{x\to 0^+} h(x, t) = 0$ for every $t > 0$. However, this is not an accident for kernel function only; it can happen for continuous measures also.

EXAMPLE 1. There exists a positive parabolic function h in R, given by a continuous measure H on ∂R, so that $\liminf_{x\to 0^+} h(x, t) = 0$ *for every t.*

In fact, we let $S = \{-\sum_{n=1}^{\infty} a_n(10)^{-10^n}: a_n = 0 \text{ or } 1\}$, H be any continuous probability measure on $\{0\} \times S$ and

$$h(x, t) = \int_{s\in S} K(x, t: 0, s)\, dH(0, s). \tag{1.1}$$

We observe that

$$\sup\{K(x, t; 0, s): t - s \geqslant x^{1/3} \text{ or } t - s \leqslant x^3\} = o(1) \qquad \text{as} \quad x \to 0^+. \tag{1.2}$$

For each $t \in S$ and positive integer m, the interval $I \equiv [t - 10^{-(1+10^m)}, t - 10^{1-10^{m+1}}]$ does not meet S. We set $x_m = 10^{-3(1+10^m)}$ and notice that if $s \notin I$, then $t - s \leqslant x_m^3$ or $t - s \geqslant x_m^{1/3}$. Hence from (1.1) and (1.2), it follows that

$$u(x_m, t) = o(1) \qquad \text{as} \quad m \to \infty.$$

Hence $\liminf_{x\to 0^+} u(x, t) = 0$ if $t \in S$. Because S is closed, $\lim_{x\to 0^+} u(x, t) = 0$ whenever $t \notin S$. This verifies our example.

In spite of this discouraging example, we may prove the following one-sided relative Fatou theorem for positive parabolic functions in R.

THEOREM 1. *Let u and h be two positive parabolic functions in R, and U and H be the Borel measures on ∂R corresponding to u and h as in* (0.3).

Then u/h has a finite one-sided parabolic limit at H-almost every point of ∂R. This limit is H-almost everywhere the Radon–Nikodym derivative of the absolutely continuous component of U with respect to H.

We say a function f in R has one-sided parabolic limit L at $(0, s)$, if for every a, $0 < a < 1$, $\lim f(x, t) = L$ as $(x, t) \to (0, s)$ inside $(ax^2 < t - s < a^{-1}x^{-2})$.

Lemma 1 below is an exercise in differential calculus; Lemma 2 is a variant of classical measure theory.

LEMMA 1. *Suppose $0 < a < 1$ and $a \leqslant b \leqslant a^{-1}$. Then for fixed s and (x, t) satisfying $t - s = bx^2$, we have*

(i) *$K(x, t; 0, \tau)$ is an increasing function of τ when $-\infty < \tau - s < (b - \frac{1}{6})\, x^2$;*

(ii) *$K(x, t; 0, \tau)$ is a decreasing function of τ when $(b - \frac{1}{6})\, x^2 < \tau - s < \infty$; and*

(iii) *$cx^{-2} \leqslant K(x, t; 0, \tau) \leqslant Cx^{-2}$ when $-x^2/6 < \tau - s < \max\{b/2, b - \frac{1}{12}\}\, x^2$, where c and C are constants depending on a only.*

LEMMA 2. *Let U and H be two positive Borel measures on the real line $-\infty < s < \infty$. Then for H-almost every s.*

$$\lim_{c \to 0+} \frac{U([s, s + c))}{H([s, s + c))} = \lim_{c \to 0+} \frac{U((s - c, s]))}{H((s - c, s]))} = \lim_{c \to 0+} \frac{U((s - c, s + c))}{H((s - c, s + c))} < \infty. \tag{1.3}$$

The limit exists and is positive for U-almost every s.

LEMMA 3. *Let $0 < a \leqslant 1$ and h be a positive parabolic function in R corresponding to a Borel measure H on ∂R. Then for H-almost every $(0, s)$ on ∂R,*

$$\liminf h(x, t) > 0 \tag{1.4}$$

as $(x, t) \to (0, s)$ inside $(ax^2 < t - s < a^{-1}x^2)$.

Proof. From Lemma 1(iii) and (0.3) one sees that

$$h(x, t) \geqslant cH(\{(0, \tau) : |s - \tau| < ax^2/6\})\, x^{-2},$$

for some constant c depending on a. Thus by Lemma 2, for H-almost every $(0, s)$, (1.4) holds. (To apply Lemma 2, we note the symmetric derivative $dH/dx > 0$ H-almost everywhere.)

Proof of Theorem 1. Let $0 < a < 1$ and $(0, s)$ be a point where (1.3) and (1.4) both hold. At such a point $(0, s)$,

$$\lim \frac{\int_0^\infty \sinh \lambda x e^{\lambda^2 t} \, d\mu(\lambda)}{h(x, t)} = 0$$

and

$$\liminf h(x, t) > 0$$

as $(x, t) \to (0, s)$ inside $(ax^2 < t - s < a^{-1}x^2)$. We may assume that u and h are in the simpler forms:

$$u(x, t) = \int_{\partial R} K(x, t; 0, s) \, dU(0, s),$$

$$h(x, t) = \int_{\partial R} K(x, t; 0, s) \, dH(0, s).$$

From Lemma 3.3 in [2] we see that in order to prove the theorem it is enough to show

$$\limsup \frac{u(x, t)}{h(x, t)} \leqslant C(a) \lim_{c \to 0^+} \frac{U((s - c, s + c))}{H((s - c, s + c))} \tag{1.5}$$

as $(x, t) \to (0, s)$ in $(ax^2 < t - s < x^2/a)$, where $C(a)$ is a constant depending on a only. And (1.5) follows from (1.3), (1.4), Lemma 1 and arguments similar to those in [2, p. 221; 14, p. 165], where ratio Fatou theorems for harmonic functions are proved.

A probabilistic proof of Theorem 1 has been communicated to us by Taylor; see [7].

2. Boundary Behavior of Heat Potentials

Suppose Ω is Dirichlet regular region for heat equation and G is the Green's function on Ω, and μ is a positive mass distribution on Ω. We say w is a heat potential on Ω given by μ if $w \not\equiv +\infty$ and

$$w(x, t) = \int_{\Omega} G(x, t; y, s) \, d\mu(y, s).$$

Basic properties of heat potential can be found in [1] or [10]. Doob proved in [1, Theorems 5.1 and 8.2] that

THEOREM E. *If w is a heat potential on Ω and P is a fixed point in Ω, then w has limit zero along almost every Brownian trajectory which has decreasing t from P to $\partial\Omega$.*

A simple consequence of Theorem 2 of Wu in [15] shows that

THEOREM F. *If w is a heat potential in R, then*

$$\lim_{x\to 0^+} w(x, t) = 0 \tag{2.1}$$

for Lebesgue-almost every t.

However, an analogous result does not hold for potentials in D; see [16] and the following example.

EXAMPLE 2. There exists a heat potential w in D, such that for every x, $0 < x < 1$,

$$\limsup_{t\to 0^+} w(x, t) = +\infty. \tag{2.2}$$

(Details follow later.)

It is known, however, that $\liminf_{t\to 0^+} w(x, t) = 0$ a.e. on ∂D, for any potential w in D; see [12, Theorem 1].

The proof of Theorem 2 in [15] uses Doob's results above, and hence uses the probability theory implicitly. In case the domain is the half-plane R, a nonprobabilistic proof can be given.

The Green's function $G(x, t; y, s)$ in R has the representation

$$\begin{aligned} G(x, t; y, s) &= \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{t-s}} e^{-(x-y)^2/4(t-s)} [1 - e^{-xy/(t-s)}], & t > s \\ &= 0 & t \leqslant s. \end{aligned} \tag{2.3}$$

LEMMA 4.

$$G(x, t; y, s) \leqslant (t-s)^{-1/2}, \tag{2.4}$$

$$G(x, t; y, s) \leqslant xy(t-s)^{-3/2}, \tag{2.5}$$

$$G(x, t; y, s) \leqslant cyx^{-2} \qquad \textit{if} \quad 0 \leqslant y \leqslant x/2, \tag{2.6}$$

$$G(10, 10; y, s) \geqslant cy \qquad \textit{if} \quad 0 \leqslant s \leqslant 1 \quad \textit{and} \quad 0 < y < 1, \tag{2.7}$$

where c's are absolute constants.

Proof. The estimates (2.4), (2.5) and (2.7) follows from (2.3) easily.

Fix (x, t), and let $v(y, s) = yx^{-2}$ on $\Omega \equiv \{(y, s)\colon 0 \leqslant y \leqslant x/2$ and $s \leqslant t\}$. Then $G(x, t; y, s)$ and $v(y, s)$ are solutions of $\partial^2 f/\partial y^2 + \partial f/\partial s = 0$ on $\Omega = \{(y, s)\colon 0 < y < x/2,\ s < t\}$, and satisfy

$$G(x, t; y, s) \leqslant cv(y, s) \tag{2.8}$$

on the boundary $\partial\Omega$. Because both G and v are bounded in Ω, by the adjoint form of the maximum principle in [4], (2.8) also holds in Ω. This gives the estimate (2.6).

Proof of Theorem F. Because of Theorem B, we may assume μ is supported on $\{(y, s)\colon 0 < y < 1$ and $0 < s < 1\}$ and assume $w(10, 10) < \infty$ and we need only show (2.1) for $0 < t < 1$.

Let $\varepsilon(q) = \int_{0<y<q, 0<s<1} G(10, 10; y, s)\, d\mu(y, s)$ for $q > 0$ and let

$$\Phi(t, r, q) = \int G(10, 10; y, \tau)\, d\mu(y, \tau)$$
$$\text{over } \{(y, \tau)\colon 0 < y < q \text{ and } |\tau - t| < r\}. \tag{2.9}$$

Let $E(q)$ be the set of t, where

$$\limsup_{r \to 0} \frac{\Phi(t, r, q)}{r} \leqslant \sqrt{\varepsilon(q)}. \tag{2.10}$$

One may show that the Lebesgue measure of $\{s\colon 0 \leqslant s \leqslant 1\} \backslash E(q)$ is at most $C\sqrt{\varepsilon(q)}$, where C is independent of q. Because $\varepsilon(q) \to 0$ as $q \to 0$, in order to show (2.1) it is sufficient to show that for small $q > 0$ and $t \in E(q)$

$$\limsup \int_{\substack{0<y<q\\ 0<s<1}} G(x, t; y, s)\, d\mu(y, s) \leqslant C\sqrt{\varepsilon(q)} \tag{2.11}$$

as $x \to 0^+$. We denote by I the integral in (2.11) and write

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &\equiv \int_{R_1} + \int_{R_2} + \int_{R_3} G(x, t; y, s)\, d\mu(y, s), \end{aligned}$$

where

$$\begin{aligned} R_1 &= \{(y, s)\colon y > x/2,\ 0 < t - s < x^2\}, \\ R_2 &= \{(y, s)\colon 0 < y < x/2,\ 0 < t - s < x^2\}, \\ R_3 &= \{(y, s)\colon t - s \geqslant x^2\}. \end{aligned}$$

For a fixed $t \in E(q)$, we write $\Phi(r) = \Phi(t, r, q)$. From (2.10), Lemma 4, definition of R_i, and elementary integration by parts, it follows that

$$
\begin{aligned}
I_1 &\leqslant c \int_{R_1} (t-s)^{-1/2} y^{-1} G(10, 10; y, s)\, d\mu(y, s) \\
&\leqslant \frac{c}{x} \int_0^{x^2} r^{-1/2}\, d\Phi(r) = \frac{c}{x} \left[\frac{\Phi(r)}{r^{1/2}} \Big|_0^{x^2} + \frac{1}{2} \int_0^{x^2} \frac{\Phi(r)}{r^{3/2}}\, dr \right] \\
&= O(1) \sqrt{\varepsilon(q)} \qquad \text{as} \quad x \to 0.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leqslant C \int_{R_2} x^{-2} G(10, 10; y, s)\, d\mu(y, s) \\
&\leqslant C x^{-2} \int_0^{x^2} d\Phi(r) \\
&= O(1) \sqrt{\varepsilon(q)} \qquad \text{as} \quad x \to 0.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\leqslant C \int_{R_3} x(t-s)^{-3/2}\, G(10, 10; y, s)\, d\mu(y, s) \\
&\leqslant C x \int_{x^2}^{1} r^{-3/2}\, d\Phi(r) = C x \left[\frac{\Phi(r)}{r^{3/2}} \Big|_{x^2}^{1} + \frac{3}{2} \int_{x^2}^{1} \frac{\Phi(r)}{r^{5/2}}\, dr \right] \\
&\leqslant C x \left[\Phi(1) + \frac{3}{2} \int_{x^2}^{1} \frac{\sqrt{\varepsilon(q)} + o(1)}{r^{3/2}}\, dr \right] \\
&= \sqrt{\varepsilon(q)}\, O(1) + o(1) \qquad \text{as} \quad x \to 0.
\end{aligned}
$$

Therefore, $I \leqslant C \sqrt{\varepsilon(q)}$ for sufficiently small $x > 0$. Thus (2.11) holds and the theorem is proved.

LEMMA 5. *There exists a potential $v(x, t)$ in D and a positive decreasing function $t = g(x)$ on $0 \leqslant x < 1$ so that*

$$v(x, g(x)) = +\infty \tag{2.12}$$

for every x satisfying $0 \leqslant x < 1$.

Proof. Let $0 \leqslant x < 1$ and $x = \sum_1^\infty \varepsilon_n(x)\, 2^{-n}$, where $\varepsilon_n(x) = 0$ or 1, be the binary expansion of x containing infinitely many digits $\varepsilon_n(x) = 0$. Let $[t]$ be the greatest integer function and observe that

$$\varepsilon_n(x) = 0 \qquad \text{if } [2^n x] \text{ is even}$$

and

$$\varepsilon_n(x) = 1 \qquad \text{if } [2^n x] \text{ is odd.}$$

We define $f(x) = -\sum_1^\infty \varepsilon_n(x)\, 4^{-n}$ and

$$I(x, t) = \int_0^1 W(x, t; y, f(y))\, dy. \tag{2.13}$$

We claim that $I(x, f(x)) = +\infty$ for $0 \leqslant x < 1$.

To see this, we fix an x and suppose $\varepsilon_k(x) = 0$ for a certain $k > 2$. We write

$$2^k x = 2p + \theta(x), \qquad 0 \leqslant \theta(x) < 1,$$

for a nonnegative integer p. Suppose $0 \leqslant y < 1$ and y can be represented as

$$2^k y = 2p + 1 + \theta(y), \qquad 0 \leqslant \theta(y) < 1.$$

Then, using $0 \leqslant \theta(x),\ \theta(y) < 1$, we have $[2^{k-1}x] = [2^{k-1}y] = p$, hence $\varepsilon_n(x) = \varepsilon_n(y)$ for $1 \leqslant n \leqslant k-1$. Therefore, when

$$2p + 1 \leqslant 2^k y < 2p + 2 \tag{2.14}$$

we have $0 < y - x \leqslant 2^{1-k}$, $\frac{2}{3}4^{-k} \leqslant f(x) - f(y) \leqslant \frac{4}{3}4^{-k}$; thus

$$W(x, f(x); y, f(y)) > C 2^k$$

for a certain absolute constant $C > 0$. The numbers y satisfying (2.14) fill an interval of length 2^{-k}; and from (2.13) the contribution to $I(x, f(x))$ exceeds $2C$. Since $\varepsilon_k(x) = 0$ infinitely often, $I(x, f(x)) = +\infty$.

The restriction of W to $D \times D$ is the Green's function on D. Let $g(x) = 1 + f(x)$ and $v(x, t) \equiv \int_0^1 W(x, t; y, g(y))\, dy = I(x, t-1)$. Thus, (2.12) follows.

Construction of Example 2. For a positive integer j, let $f_j(x) = f(x)/2^j$, $I_j(x, t) = \int_0^1 W(x, t; y, f_j(y))\, dy/2^j$ and $v_j(x, t) = I_j(x, t - 2^{-j})$. The reasoning of Lemma 5 shows that $v_j(x, (1 + f(x))/2^j) = +\infty$. Letting

$$w(x, t) = \sum_{j=1}^{\infty} v_j(x, t)$$

we may obtain (2.2) and $w(0, 10) < +\infty$ easily. This gives Example 2.

3. Heat Capacity

For rigorous definitions of heat capacity, adjoint heat capacity, polar sets and adjoint polar sets, the reader is referred to [1, 11]. The following two theorems are due to Watson [11, Theorems 7 and 9].

THEOREM G. *Let Z be a subset of $\{-\infty < x < \infty\} \times \{-\infty < t < \infty\}$. Then Z has heat capacity zero $\Leftrightarrow Z$ has adjoint heat capacity zero $\Leftrightarrow Z$ is heat polar $\Leftrightarrow Z$ is adjoint heat polar.*

THEOREM H. *On $\{(x, 0): -\infty < x < \infty\}$, heat capacity and the linear Lebegue measure have the same null-sets.*

On the other hand, we can prove the following.

THEOREM 2. *If E is a Borel set on $\{(0, t): -\infty < t < \infty\}$, then E has heat capacity zero if and only if E has classical $\frac{1}{2}$-capacity zero.*

Proof. Suppose E is compact. By a modification of [11, Lemma 4],

$$\text{heat-cap}(E) = \sup \left\{\mu(E): \int_E W(x, t; 0, \tau)\, d\mu(0, \tau) \leqslant 1 \text{ on } R^2, \right.$$
$$\left. \mu \text{ is supported on } E \text{ and } \mu \geqslant 0 \right\}. \quad (3.1)$$

From (0.1) and its adjoint form we obtain

$$\begin{aligned} W(0, t; 0, \tau) + W^*(0, t; 0, \tau) &= |t-\tau|^{-1/2} && \text{if} \quad t \neq \tau \\ &= 0 && \text{if} \quad t = \tau. \end{aligned} \quad (3.2)$$

Suppose E is compact and has zero $\frac{1}{2}$-capacity. Then there exists a positive measure μ on E so that $P(0, t) = \int_E |t-\tau|^{-1/2}\, d\mu(0, \tau)$ satisfies $P(0, t) = +\infty$ on E but $P \not\equiv +\infty$. Let $S = \{(0, t): \mu\{(0, t)\} > 0\}$ and let $I(t) = \int_E W(0, t; 0, \tau)\, d\mu(0, \tau)$ and

$$I^*(t) = \int_E W^*(0, t; 0, \tau)\, d\mu(0, \tau);$$

moreover, let $F = \{(0, t) \in E: I(t) = +\infty\}$ and

$$F^* = \{(0, t) \in E: I^*(t) = +\infty\}.$$

We observe that $F \cup F^* \cup S = E$ and that F is heat polar, F^* is adjoint heat polar and S is countable. By Theorem G, we conclude that E has heat capacity zero. Suppose E has zero heat capacity, then naturally E has zero $\frac{1}{2}$-capacity because $W(0, t; 0, \tau) \leqslant |t-\tau|^{-1/2}$. One can easily extend these to Borel sets by the definition of capacity. This completes the proof.

Let $\alpha > 0$,

$$\begin{aligned} h(t) &= t^{-\alpha} && \text{if} \quad t > 0 \\ &= 0 && \text{if} \quad t \leqslant 0 \end{aligned}$$

and

$$h^*(t) = 0 \qquad \text{if} \quad t \geqslant 0$$
$$= (-t)^{-\alpha} \qquad \text{if} \quad t < 0.$$

Let E be a compact subset of $\{-\infty < t < \infty\}$ and define

$$\text{h-cap}(E) = \sup \left\{ \mu(E) \colon \int_E h(t-\tau)\, d\mu(\tau) \leqslant 1 \text{ on } R^2, \mu \geqslant 0, \right.$$
$$\left. \text{and } \mu \text{ is supported on } E \right\}$$

and define h*-cap(E) similarly.

Question. Are h-capacity zero, h*-capacity zero and α-capacity zero equivalent? When $\alpha = \frac{1}{2}$, they are equivalent by Theorem 2 whose proof depends heavily on Theorem G. It seems interesting to investigate their relationship in the case $\alpha \neq \frac{1}{2}$, when $h(t)$ and the fundamental solution W of heat equation are unrelated. However the authors have no conjecture on this point.

How large can a set of heat capacity zero be? It can be much larger than we expect. Theorem G and Lemma 5 give the following:

EXAMPLE 3. There exists a set Z of capacity zero, where Z is the graph $\{(x, g(x))\colon 0 \leqslant x < 1\}$ of some decreasing function $t = g(x)$; and the projection $\{(0, g(x))\colon 0 \leqslant x < 1\}$ *is of Hausdorff dimension* $\frac{1}{2}$ and with positive $\frac{1}{2}$-dimensional Hausdorff measure.

If E is of heat capacity zero, then almost every Brownian trajectory with decreasing t, from a point will avoid E [1, Theorem 9.1]. Therefore, the function $t = g(x)$ in Example 3 cannot be continuous and so is certainly not smooth. In fact the set Z is totally disconnected. However, if we are willing to discard a small portion of the interval (0, 1), we can make $t = g(x)$ nearly C^2.

EXAMPLE 4. *Let* $\varepsilon > 0$. *There exists a function* $t = l(x)$ *on* $0 < x < 1$, *which is of class* $C^{2-\alpha}$, *for any* α, $0 < \alpha < 1$, *and a set* $E \subseteq \{(x, l(x))\colon 0 < x < 1\}$, *so that* E *has zero heat capacity*; *nevertheless* $\{(x, 0)\colon (x, l(x)) \in E\}$ *has linear Lebesgue measure greater than* $1 - \varepsilon$.

Question. We do not know if this can happen on a curve of class C^2, or even class C^∞.

To construct the example, it suffices to find a function $l(x)$ in each class $C^{2-\alpha}$, so that the Lebesgue measure $m\{x\colon l(x) = f(x)\} > 1 - \varepsilon$, for the function f in Lemma 5.

Let r be a number between 0 and 1/4. Remove from $[0, 1]$ all the intervals $\bigcup_{n=1}^{\infty} \bigcup_{p=0}^{\infty} (p2^{-n} - rn^{-2}2^{-n},\ p2^{-n} + rn^{-2}2^{-n})$ and obtain a subset $S \subseteq [0, 1]$ of measure $m(S)$, and

$$1 - m(s) = O(r). \tag{3.3}$$

For any fixed x in S and any positive integer p, we have $|2^n x - p| \geqslant rn^{-2}$. Suppose $0 \leqslant y < 1$ and $\varepsilon_n(x) \neq \varepsilon_n(y)$ for a certain $n \geqslant 1$. Then $[2^n x] \neq [2^n y]$; hence there is a positive integer p, so that

$$2^n x < p \leqslant 2^n y \qquad \text{or} \qquad 2^n y < p \leqslant 2^n x.$$

In either case, $|2^n x - 2^n y| \geqslant |2^n x - p| \geqslant rn^{-2}$, or $|x - y| \geqslant rn^{-2}2^{-n}$.

We recall that $f(x) = -\sum_1^\infty \varepsilon_k(x)\, 4^{-k}$ when $x = \sum_1^\infty \varepsilon_k(x)\, 2^{-k}$. Therefore, whenever $x \in S$ and $0 \leqslant y < 1$, we obtain

$$|f(x) - f(y)| \leqslant 4^{-n+1} \leqslant A(r)\, |x - y|^2 \log^4(|x - y|^{-1} + e) \tag{3.4}$$

for some constant A depending only on r.

We set $\psi(t) = t^2(3 - 2t)$, so $\psi'(0) = \psi'(1) = 0$ and $|\psi''(t)| \leqslant 6$ for $0 \leqslant t \leqslant 1$. We define $l(x) = f(x)$ for $x \in S$; and on an interval (a, b) contiguous to S, we set

$$l(x) = l(a) + (l(b) - l(a))\, \psi\left(\frac{t - a}{b - a}\right).$$

Then on $[a, b]$, $l'(a) = l'(b) = 0$; and when $a < x_2 < x_1 < b$,

$$\begin{aligned} |l'(x_1) - l'(x_2)| &\leqslant 6\, |l(b) - l(a)|\, (x_1 - x_2)(b - a)^{-2} \\ &\leqslant 6A(r) \log^4((b - a)^{-1} + e)(x_1 - x_2) \\ &\leqslant 6A(r) \log^4(|x_1 - x_2|^{-1} + e)(x_1 - x_2), \end{aligned} \tag{3.5}$$

by (3.4).

When $0 < x_1, x_2 < 1$ but separated by an element of S, we may obtain a similar bound for $|l'(x_1) - l'(x_2)|$ by combining (3.4) and (3.5). Therefore $l \in C^{2-\alpha}$ for any $0 < \alpha < 1$. In view of (3.3), the construction is complete. The idea behind the construction is from the Whitney Extension Theorem [8. p. 170]. This completes Example 4.

4. Existence and Uniqueness for Elementary Regions

Let $\tau(x)$ be continuous for $-\infty < x < \infty$ and Ω_τ be the region $\{(x, t): t > \tau(x)\}$.

THEOREM 3. *Suppose u is a subparabolic function in Ω_τ* (see [1] or [10] for definition) *satisfying*

$$\limsup u \leqslant 0 \qquad \text{on } \partial\Omega_\tau$$

and

$$u(x, t) \leqslant \psi(|x|) \tag{4.1}$$

with $\psi > 0$ and $\psi(r) = o(r)$ as $r \to +\infty$. Then $u \leqslant 0$ everywhere in Ω.

Proof. Let $r > 0$ and let

$$V_r = \Omega_\tau \cap \{(x, t)\colon 0 < x < r,\ t < \tau(0)\}.$$

We observe that on the part of the boundary where $t < \tau(0)$,

$$u(x, t) \leqslant x\psi(r)/r. \tag{4.2}$$

By the maximum principle for parabolic functions [4, p. 152] and the definition of a subparabolic function [1, p. 219], (4.2) holds in V_r also. For a fixed $(x, t) \in \Omega_\tau$ with $x > 0$, $t < \tau(0)$, we let $r \to \infty$ in (4.2) and conclude $u(x, t) \leqslant 0$. When $x < 0$, we argue similarly and obtain

$$u(x, t) \leqslant 0 \qquad \text{in } \Omega_\tau \cap \{(x, t)\colon t < \tau(0)\}. \tag{4.3}$$

Now fix $a < \tau(0)$ and let

$$U_r = \Omega_\tau \cap \{(x, t)\colon |x| < r,\ t > a\}.$$

We observe that on ∂U_r,

$$u(x, t) \leqslant \psi(r)\, r^{-2}(x^2 + 2(t - a)), \tag{4.4}$$

and the majorant is parabolic. By the maximum principle, (4.4) holds also in U_r. For any fixed (x, t) with $t > a$, we let $r \to \infty$ in (4.4) and conclude

$$u(x, t) \leqslant 0 \qquad \text{in } \Omega_\tau \cap \{(x, t)\colon t > a\}. \tag{4.5}$$

The theorem follows from (4.3) and (4.5).

THEOREM 4. *Suppose $\tau(x) \geqslant (m - \frac{1}{2})\, x^2$ for a certain $m > 0$ and u is a subparabolic function on Ω_τ. If*

$$\limsup u \leqslant 0 \qquad \textit{on } \partial\Omega_\tau$$

and

$$u(x, t) \leqslant \psi(|x|) \tag{4.6}$$

with $\psi \geqslant 0$ *and* $\psi(r) = o(r^2)$ *as* $r \to +\infty$, *then* $u \leqslant 0$ *everywhere in* Ω_τ.

Proof. On the region $\Omega_\tau \cap \{(x, t): |x| < r\}$, we have the inequality

$$u(x, t) \leqslant m^{-1} r^{-2} \psi(r)(t + \tfrac{1}{2}x^2), \tag{4.7}$$

because the inequality holds on the boundary and $t + \frac{1}{2}x^2$ is a parabolic function. For a fixed (x, t), letting $r \to +\infty$ in (4.7), we get $u(x, t) \leqslant 0$, and the proof is complete.

A standard deduction gives the maximum principle and the uniqueness theorems for parabolic functions satisfying growth conditions (4.1) or (4.6) in regions described in Theorems 3 and 4, respectively.

To see at once the condition "$\psi(r) = o(r)$ as $r \to +\infty$" cannot be replace by "$\psi(r) = O(r)$ as $r \to +\infty$" in Theorem 3, and the region Ω in Theorem 4 is nearly the best possible, we construct the following example.

EXAMPLE 5. *In the region* Ω_τ *with* $\tau(x) = -x^2[\log(e + |x|)]^3$, *there is a parabolic function* $u(x, t)$ *such that*

$$\begin{aligned} &u(x, t) \leqslant x && \text{in } \Omega_\tau, \\ &u(x, t) \leqslant 0 && \text{on } \partial\Omega_\tau, \end{aligned} \tag{4.8}$$

and

$$u > 0 \qquad \text{at some point of } \Omega_\tau. \tag{4.9}$$

Let u_n be bounded parabolic function in Ω_τ, which has boundary value 1 on $\partial\Omega_\tau \cap \{(x, t): x > 2^n\}$ and 0 on $\partial\Omega_\tau \cap \{(x, t): x < 2^n\}$ and $0 \leqslant u_n \leqslant 1$. (The existence follows from Theorem 5.) We define

$$u(x, t) = x - 4 - \sum_1^\infty 2^{n+1} u_n(x, t). \tag{4.10}$$

Clearly $u(x, t) < 0$ if $x < 4$. When $x \geqslant 4$, we choose $n \geqslant 2$ so that $2^{n+1} \geqslant x > 2^n$; at the point $(x, \tau(x))$, $2^{n+1} u_n(x, t) \geqslant 2^{n+1} \geqslant x$, whence $u(x, \tau(x)) < -4$. This proves (4.8). Once it is shown that the series in (4.10) converges uniformly on each compact subset of Ω_τ and that $\sup u > 0$, the example will be complete.

We define an auxiliary function $w(x, t)$ for $x > 0$, $t = 0$:

$$w(x, t) = \varphi(x/\sqrt{t}),$$

where $\varphi(s) \equiv \pi^{-1/2} \int_0^s \exp(-u^2/4)\,du$. Then w is parabolic and $w(x, 0^+) \equiv 1$, $w(0^+, t) \equiv 0$ and $w(x, t) < x/\sqrt{t}$.

On the domain $\Omega_\tau \cap \{(x, t)\colon x > 0,\ 0 \geqslant t \geqslant \tau(2^n)\}$, the maximum principle derived from Theorem 3 shows that

$$u_n(x, t) \leqslant w(x, t - \tau(2^n)) < x(t - \tau(2^n))^{-1/2}.$$

As soon as $0 \geqslant t \geqslant -2^n$, we have

$$2^{n+1} u_n(x, t) \leqslant 2^{n+2} x\,|\tau(2^n)|^{-1/2} = O(n^{-3/2}\,|x|); \tag{4.11}$$

of course the same is true for $x < 0$. This proves the uniform convergence on compact subsets of $\Omega_\tau \cap \{(x, t)\colon t \leqslant 0\}$.

Because u_n is bounded and parabolic for $t > 0$, by (4.11),

$$\begin{aligned} 2^{n+1} u_n(x, t) &\leqslant 2^{n+1} \int_{-\infty}^{\infty} |y|\,|\tau(2^n)|^{-1/2}\, W(x, r; y, 0)\,dy \\ &\leqslant O(n^{-3/2}) \int_{-\infty}^{\infty} |y|\, W(x, t; y, 0)\,dy \\ &= O(n^{-3/2}) \cdot (|x| + \sqrt{t}) \qquad \text{for} \quad t \geqslant 0. \end{aligned}$$

This proves the uniform convergence of the series on compact subsets of $\Omega_\tau \cap \{(x, t)\colon t \geqslant 0\}$.

To verify that $u(x, 0) > 0$ for large $x > 0$, we observe

$$\begin{aligned} u(x, 0) &\geqslant x - 4 - \sum_1^{\infty} 2^{n+1} \min(1, |x|\,|\tau(2^n)|^{-1/2}) \\ &\geqslant x - 4 - 2 \sum_1^{\infty} \min(2^n, |x|\, n^{-3/2}) \\ &= x - 4 - o(x) \qquad \text{as} \quad x \to \infty. \end{aligned}$$

Thus $\sup u(x, 0) > 0$. This completes the example.

THEOREM 5. *Let Ω be a domain in E^2. A point $(x_0, t_0) \in \partial\Omega$ is a regular boundary point for the heat equation if $E^2 - \Omega$ contains a line segment $\{(x_0, S)\colon t_0 - \delta \leqslant s \leqslant t_0\}$. In particular the region $\{(x, t)\colon t > f(x)\}$ for some continuous $f(x)$, $-\infty < x < \infty$, is Dirichlet regular for the heat equation.*

Proof. Let $v(x, t) = \int_0^\delta W(x, t; x_0, t_0 - s)\,ds$. Then v is parabolic outside $\{(x_0, s)\colon t_0 - \delta \leqslant s \leqslant t_0\}$ and attains a strict absolute maximum at (x_0, t_0). Therefore $v(x_0, t_0) - v(x, t)$ is a barrier at (x_0, t_0). Thus (x_0, t_0) is a regular point by [10, Theorem 34].

Question. We do not know whether Theorem 4 is valid when $m = 0$ or $m < 0$, and we do not know whether Example 5 can be done for the region $(t > -\frac{1}{2}x^2)$.

5. Parabolic Measures

If Ω is Dirichlet regular for heat equation, for a fixed point $(y, s) \in \Omega$ the parabolic measure of a Borel set $E \subseteq \partial\Omega$ at (y, s), denoted by $\omega^{(y,s)}(E)$, is defined to be the value at (y, s) of the solution of the heat equation on Ω with boundary value 1 on E and 0 on $\partial\Omega \backslash E$, in the Brelot–Perron–Wiener sense. We define the adjoint parabolic measure $\omega^{*(y,s)}(E)$ similarly for the adjoint heat equation. We say $E \subseteq \partial\Omega$ has parabolic measure zero, without referring to a specific point (y, s), if $\omega^{(y,s)}(E) = 0$ for every $(y, s) \in \Omega$.

It is well known that on the boundary of R (or D), sets of parabolic measure zero are exactly those of Lebesgue measure zero. In [5, 15], the following results were proved, respectively.

THEOREM I. *Suppose $x = f(t)$ is a* Lip $\frac{1}{2}$ *function for $-\infty < t < \infty$ $\Omega = \{(x, t)\colon x > f(t)\}$ and E is a subset of $\partial\Omega$ whose projection $\{t\colon (x, t) \in E\}$ has Lebesgue measure zero. Then E is composed of two parts, one with parabolic measure zero, and the other with adjoint parabolic measure zero.*

THEOREM J. *There exists a* Lip $\frac{1}{2}$ *function $x = f(t)$, so that the parabolic measure $\omega^{(x,t)}$, the adjoint parabolic measure $\omega^{*(y,s)}$ and the projection measure p on $\partial\Omega$ $(\Omega \equiv \{(x, t)\colon x > f(t)\})$ are mutually singular, where (x, t), (y, s) are any points in Ω and $p(E) =$ Lebesgue measure of $\{t\colon (x, t) \in E\}$.*

Question. Can $x = f(t)$ in Theorem J become Lip α for $\alpha > \frac{1}{2}$?

In this section, we study the relation between Lebesgue measure and parabolic measure on boundary of a region given by $\{(x, t)\colon t > \tau(x)\}$.

We first write down the solution of the Dirichlet problem for the half-plane $H_m = \{(x, t)\colon\ t > mx\}$. Similar formulas are presented in [9, pp. 447–448].

THEOREM 6. *To each bounded continuous function f on ∂H_m the unique bounded solution of the Dirichlet problem for heat equation in H_m is*

$$F(x, t) \equiv \int_{-\infty}^{\infty} (t - mx)(t - my)^{-1}\, W(x, t; y, my)\, f(y, my)\, dy. \tag{5.1}$$

Proof. The uniqueness follows from Theorem 3. The proof of the existence of a bounded solution in the form (5.1) is equivalent to showing

$$(t - mx)(t - my)^{-1}\, W(x, t; y, my)\, dy \tag{5.2}$$

is the parabolic measure on ∂H_m at a point (x, t) in H_m. To this end, we observe that the integrand in (5.2), being equal to $(1 + 2m\,\partial/\partial x)$ $W(x, t; y, my)$, is parabolic in (x, t); it is positive because $W = 0$ unless $t > my$. To show formula (5.2) is the parabolic measure we need only to show

$$g(x, t) \equiv \int_{-\infty}^{\infty} (t - mx)(t - my)^{-1}\, W(x, t; y, my)\, dy = 1 \tag{5.3}$$

for each $(x, t) \in H_m$. This is known for $m = 0$.

By a change of variable, we observe that $g(x, t)$ is a function $\tilde{g}$ of $t - mx$ alone. Because $g(x, t)$ is parabolic, simple calculation shows that $\tilde{g}(z) = A + Be^{m^{-2}z}$ for some constants A, B; thus

$$g(x, t) = A + Be^{m^{-2}(t - mx)}.$$

Hence to show (5.3), we need only to show

$$g(0, t) = O(1) \qquad \text{as} \quad t \to +\infty \tag{5.4}$$

and

$$\lim_{t \to 0+} g(0, t) = 1. \tag{5.5}$$

We assume as we may that $m > 0$. To show (5.4), we write

$$\begin{aligned} g(0, t) &= \int_{\{my < t\}} t(t - my)^{-1}\, W(0, t; y, my)\, dy \\ &= \int_{\{|my| \leqslant t/2\}} + \int_{\{|my| > t/2\}}. \end{aligned} \tag{5.6}$$

Making substitutions $y = u\sqrt{t}$ in the first integral and $t - my = t^2 u$ in the second integral, we may obtain $g(0, t) = O(1)$ as $t \to +\infty$.

To show (5.5), we make the substitution $t - my = t^2 v$ in (5.6) to obtain

$$g(0, t) = (4\pi)^{-1/2}\, m^{-1} \int_0^{\infty} v^{-3/2} \cdot \exp[-(1 - tv)^2\, (4m^2 v)^{-1}]\, dv.$$

For $0 < t < 1/2$, the integrand is dominated by the integrable function $m^{-1} v^{-3/2}$ for $v > 1$, and by the integrable function $m^{-1} v^{-3/2}$ $\exp(-[(8m^2 v)^{-1}])$ for $0 < v < 1$. Hence the limit, as $t \to 0^+$, is

$$g(0+, t) = (4\pi)^{-1/2}\, m^{-1} \int_0^{\infty} v^{-3/2} \exp(-[(4m^2 v)^{-1}])\, dv,$$

which has value 1 by a change of variable $v = w^{-2}$. This proves (5.5), and completes the proof of our theorem.

Next we turn to the region $\Omega_\tau \equiv \{(x, t)\colon t > \tau(x)\}$ and prove

THEOREM 7. *Suppose $\tau \in C^2(-\infty, \infty)$. Let (a, b) be an interval on which $\tau' \geqslant \delta > 0$ for some δ and let I be the arc $\{(x, \tau(x))\colon a \leqslant x \leqslant b\}$. Then on I, sets of parabolic measure zero with respect to the region Ω_τ are exactly those sets of length zero.*

We first define a *parametrix* for the parabolic measure on I. At each $(y_0, \tau(y_0)) \in I$, we use the density of the parabolic measure for the half-plane $t > \tau(y_0) + (x - y_0)\,\tau'(y_0)$ from Theorem 6, that is, we define

$$\Gamma(x, t; y_0, \tau(y_0)) \equiv (t - \tau(y_0) - \tau'(y_0)(x - y_0))(t - \tau(y_0))^{-1} \cdot W(x, t; y_0, \tau(y_0)). \tag{5.7}$$

This is a linear combination of $W(x, t; y_0, \tau(y_0))$ and $\partial W(x, t; y_0, \tau(y_0))/\partial y$ and is parabolic in (x, t) when $(x, t) \neq (y_0, \tau(y_0))$. From Taylor's formula with two derivatives we obtain

LEMMA 6. *$\Gamma(x, \tau(x); y_0, \tau(y_0))$ is bounded if $(y_0, \tau(y_0)) \in I$, $-\infty < x < \infty$ and $x \neq y_0$.*

LEMMA 7. *Suppose that $f(x)$ is continuous and vanishes off the interval (a, b). We claim that*

$$\lim \int_a^b \Gamma(x, t; y, \tau(y))\, f(y)\, dy = f(y_0) + \int_a^b \Gamma(y_0, \tau(y_0); y, \tau(y))\, f(y)\, dy \tag{5.8}$$

as (x, t) converges to $(y_0, \tau(y_0))$.

Proof. This is a standard deduction if $y_0 < a$ or $y_0 > b$; we shall prove (5.8) for $a < y_0 < b$, but the proof holds for end points as well.

First we write a decomposition

$$\Gamma(x, t; y, \tau(y)) \equiv \tilde{\Gamma} + \tilde{\tilde{\Gamma}},$$

where

$$\tilde{\Gamma} = (t - \tau(x))(t - \tau(y))^{-1}\, W(x, t; y, \tau(y))$$

and

$$\tilde{\tilde{\Gamma}} = (\tau(x) - \tau(y) - \tau'(y)(x-y))(t-\tau(y))^{-1} W(x, t; y, \tau(y)).$$

As $(x, t) \to (y_0, \tau(y_0))$, $\tilde{\Gamma}$ and $\tilde{\tilde{\Gamma}}$ have limits except when $y = y_0$. We shall prove $\tilde{\tilde{\Gamma}}(x, t; y, \tau(y))$ is uniformly bounded in $L^{5/4}$ $(a < y < b)$ for all (x, t); so we can use uniform integrability to obtain

$$\lim \int_a^b \tilde{\tilde{\Gamma}}(x, t; y, \tau(y))\, dy = \int_a^b \Gamma(y_0, \tau(y_0); y, \tau(y))\, dy$$

as $(x, t) \to (y_0, \tau(y_0))$ because $\Gamma = \tilde{\tilde{\Gamma}}$ at $(y_0, \tau(y_0); y, \tau(y))$.

To show the uniform $L^{5/4}$ boundedness, we observe that

$$\int_a^b |t - \tau(y)|^{-7/8}\, dy \leqslant C(\delta, b-a) \tag{5.9}$$

by using $\tau(x)$ as a variable of integration. We also observe that $\tilde{\tilde{\Gamma}} = O(|x-y|^2)\,|t-\tau(y)|^{-3/2}$. On the set of x defined by $|x-y|^2 \leqslant |t-\tau(y)|^{4/5}$, we have $\tilde{\tilde{\Gamma}} = O(|t-\tau(y)|^{-7/10})$. On the set of x defined by $|x-y|^2 > |t-\tau(y)|^{4/5}$, we have $\tilde{\tilde{\Gamma}} = O(1)$. From (5.9), the uniform $L^{5/4}$-boundedness follows.

To show

$$\lim \int_a^b \tilde{\Gamma}(x, t; y, \tau(y))\, f(y)\, dy = f(y_0) \tag{5.10}$$

as $(x, t) \to (y_0, \tau(y_0))$, we let $t - \tau(x) = \lambda$ be a small positive number. The equation $\tau(\bar{y}) = t$ has a solution in (a, b). In the case $t - \tau(y) \geqslant \lambda^{1/2}$,

$$|\tilde{\Gamma}| \leqslant (t - \tau(x))(t - \tau(y))^{-3/2} \leqslant \lambda^{1/4}. \tag{5.11}$$

In the case $0 < t - \tau(y) = \tau(\bar{y}) - \tau(y) < \lambda^{1/2}$. we have $|\bar{y} - y| = O(\lambda^{1/2})$. By Taylor's formula, we obtain

$$\begin{aligned} \tau(\bar{y}) - \tau(y) &= \tau'(\bar{y})(\bar{y} - y)(1 + O(|\bar{y} - y|)), \\ |\tau(\bar{y}) - \tau(y)|^{-3/2} &= |\tau'(\bar{y})(\bar{y} - y)|^{-3/2}\,(1 + O(|\bar{y} - y|)) \\ &= |\tau'(\bar{y})(\bar{y} - y)|^{-3/2}\,(1 + O(\lambda^{1/2})). \end{aligned}$$

For the exponential occurring in $W(x, t; y, \tau(y))$, we observe the expansions

$$\begin{gathered} |t - \tau(y)|^{-1} = |\tau(\bar{y}) - \tau(y)|^{-1} = |\tau'(\bar{y})(\bar{y} - y)|^{-1} + O(1), \\ |x - y| = O(|\tau(x) - \tau(y)|) = O(|t - \tau(x)| + |t - \tau(y)|) = O(\lambda^{1/2}), \\ |x - y|^2\,|t - \tau(y)|^{-1} = |x - y|^2\,|\tau'(\bar{y})(\bar{y} - y)|^{-1} + O(\lambda^{1/2}). \end{gathered}$$

Therefore

$$\begin{aligned}\tilde{\Gamma}(x, t; y, \tau(y)) &= (t-\tau(x))\,|\tau'(\bar{y})(\bar{y}-y)|^{-3/2}\,(4\pi)^{-1/2}\,(1+O(\lambda^{1/2}))\\ &\quad\cdot \exp[-\tfrac{1}{4}|x-y|^2\,|\tau'(\bar{y})(\bar{y}-y)|^{-1}] + O(\lambda^{1/2})\\ &= |\tau'(\bar{y})(\bar{y}-x)|\,|\tau'(\bar{y})(\bar{y}-y)|^{-3/2}\,(4\pi)^{-1/2}\\ &\quad\cdot \exp\left[-\frac{(x-y)^2}{4\,|\tau'(\bar{y})(\bar{y}-y)|}\right](1+O(\lambda^{1/2}))\\ &= \Gamma(x, t; y, t-\tau'(\bar{y})(\bar{y}-y))(1+O(\lambda^{1/2})).\end{aligned}$$

We recall that the Γ term above is the density of the parabolic measure on the line tangent to $t=\tau(x)$ through $(\bar{y}, \tau(\bar{y}))$. Hence by Theorem 6

$$f(y_0) = \int_{-\infty}^{\infty} \Gamma(x, t; y-\tau'(\bar{y})(\bar{y}-y))\, f(y)\, dy + o(1).$$

Combining (5.11), (5.12) and an estimate similar to (5.11) for $\Gamma(x, t; y, t-\tau'(\bar{y})(\bar{y}-y))$, we conclude (5.10) and complete the proof.

For each fixed $(y, \tau(y))$ in I, let $\Gamma_0(x, t; y, \tau(y))$ be the bounded, parabolic function in Ω_τ that has boundary value $\Gamma(z, \tau(z); y, \tau(y))$ at $(z, \tau(z))$, and let

$$\Gamma_1 = \Gamma - \Gamma_0. \tag{5.13}$$

THEOREM 8. *Suppose $\tau \in C^2(-\infty, \infty)$. Then $\Gamma_1(x, t; y, \tau(y))\, dy$ is the parabolic measure on I with respect to the region Ω_τ at (x, t); that is, given a continuous function $f(x)$ which vanishes off (a, b), the solution of the Dirichlet problem for heat equation in Ω_τ with boundary value $f(x)$ at $(x, \tau(x))$ is given by*

$$\int_a^b \Gamma_1(x, t; y, \tau(y))\, f(y)\, dy.$$

Moreover, $\Gamma_1(x, t; y, \tau(y))$ is a continuous function of all variables for $(x, t) \in \Omega_\tau$ and $(y, \tau(y)) \in \partial\Omega_\tau$.

Proof. In view of (5.8), the first statement is just a matter of interchanging integration and limit.

To show the continuity of $\Gamma(x, t; y, \tau(y))$ in (x, t, y), we recall the formula $\Gamma = W(x, t; y, \tau(y)) + 2\tau'(y)(\partial W(x, t; y, \tau(y))/\partial x)$. Now $W(x, t; y, s)$ is analytic except when $(x, t) = (y, s)$, and this gives the continuity.

To prove that $\Gamma_0(x, t; y, \tau(y))$ is continuous, the essential point is to show that $\sup_{(x,t)} |\Gamma_0(x, t; y, \tau(y)) - \Gamma_0(x, t; \bar{y}, \tau(\bar{y}))|$ is small when $a \leqslant y < \bar{y} \leqslant b$ and $|y-\bar{y}|$ is small. By the maximum principle, stated implicitly in Theorem 3, the above supremum is just the supremum over the boundary, $\sup_x |\Gamma(x, \tau(x); y, \tau(y)) - \Gamma(x, \tau(x); \bar{y}, \tau(\bar{y}))|$. From Taylor's formula we know that this difference is at most $C|y-x|^{3/2}+C|\bar{y}-x|^{3/2}$. To make the difference $<\varepsilon$, we have only to consider triplets $(x, y, \bar{y})$ at which either $|y-x|>\delta$ or $|\bar{y}-x|>\delta$, with $\delta=(\varepsilon/2c)^{2/3}>0$. When $|y-\bar{y}|<\delta/2$, we have in either case $|y-x|>\delta/2$ and $|\bar{y}-x|>\delta/2$. But $W(x, t; y, \tau(y))=W(x-y; y-\tau(y); 0, 0)$ and $W(x, t; 0, 0)$ is uniformly continuous and uniformly bounded on each set $|x|>\delta/2$; the same applies to $\partial W(x, t; 0, 0)/\partial x$; because τ' is bounded and continuous, this is sufficient to make the difference small. This proves the continuity of Γ_1.

Proof of Theorem 7. It follows from Theorem 8 that sets of length zero are of parabolic measure zero.

For each $(y, \tau(y))$ on I, $\Gamma(x, t; y, \tau(y))$ becomes unbounded as (x, t) approaches $(y, \tau(y))$ in a certain parabola with vertex at $(y, \tau(y))$; an explicit construction is contained in the proof of Theorem 9. The density $\Gamma_1(x, t; y, \tau(y))$ therefore becomes positive at certain (x, t) near $(y, \tau(y))$. Inasmuch as $\Gamma_0=\Gamma-\Gamma_1$ is continuous for this (x, t), $\Gamma_1(x, t, z, \tau(z))$ has a positive lower bound for $(z, \tau(z))$ in a neighborhood of $(y, \tau(y))$. This shows that a set of positive length has positive parabolic measure at a certain point (x, t). This completes the proof.

It is our intention to extend Theorem 7 to arcs on which $\tau' \geqslant \delta > 0$ on some interval $(a-\varepsilon, b+\varepsilon)$, but only one continuous derivative is assumed. We do this by writing an integral equation for the parabolic density for the smoother regions considered above and showing that the solution can be bounded a priori by quantities depending on the derivative τ' alone.

Let (x, t) be fixed for the moment and let $\lambda(y)\,dy=\Gamma_1(x, t; yt, \tau(y))\,dy$ be the parabolic measure on I. We recall from (5.13) that

$$\lambda(y)-\Gamma(x, t; y, \tau(y)) \tag{5.14}$$

is the bounded parabolic function in Ω_τ whose boundary value at $(z, \tau(z))$ is $-\Gamma(z, \tau(z); y, \tau(y))$.

We observe that

$$\sup\{|\Gamma(z, \tau(z); y, \tau(y))|\colon a \leqslant y \leqslant b,\ z \geqslant b \text{ or } z \leqslant a\} \leqslant C, \tag{5.15}$$

$$|\Gamma(z, \tau(z); y, \tau(y))| \leqslant C\,|z-y|^{-1/2}, \tag{5.16}$$

and

$$|\Gamma(x, t; y, \tau(y))| \leqslant C_1. \tag{5.17}$$

Here, and later C's depend only on a, b, ε and τ' on $(a-\varepsilon, b+\varepsilon)$ but not on τ''; C_1 depends also on the distance from (x, t) to $\partial\Omega_\tau$. From (5.14) and (5.15) we can write down the integral equation

$$\lambda(y)=\Gamma(x, t; y, \tau(y))-\int_a^b \Gamma(z, \tau(z); y, \tau(y))\,\lambda(z)\,dz+C. \tag{5.18}$$

Because

$$\int_a^b \lambda(z)\, dz \leqslant 1,$$

we obtain by Minkowski's inequality

$$\lambda(y) = \Gamma(x, t; y, \tau(y)) + g(x, t; y) \tag{5.19}$$

with

$$\int_a^b |g(x, t; y)|^{3/2}\, dy \leqslant C. \tag{5.20}$$

Before proceeding, we look at the operator T,

$$Th(x) = \int_a^b h(z)\,|z - x|^{-1/2}\, dz, \qquad a < x < b$$

defined for functions $h \in L^1(a, b)$. We see easily

$$\| Th \|_{3/2} \leqslant C \| h \|_1$$

and

$$\| Th \|_\infty \leqslant C \| h \|_{5/2}.$$

Interpolation [17, p. 112, Marcinkiewicz's theorem] between these inequalities gives

$$\| Th \|_3 \leqslant C \| h \|_{10/7} \leqslant C \| h \|_{3/2}.$$

Applying (5.16), (5.17), (5.19), (5.20) and the convolution method in the last paragraph iteratively to the integral equation (5.18), we may conclude $\int_a^b |\lambda(y)|^{3/2}\, dy$, $\int_a^b |\lambda(y)|^3\, dy$, $\|\lambda\|_\infty$ are bounded by constant C_1 depending on τ' on (a, b) but not on τ''. If we approximate a C^1 curve τ by C^2 curves whose derivatives on $[a, b]$ have comparable upper and lower bounds as τ', we see that the parabolic measure on I is absolutely continuous with respect to the Lebesgue measure dy on I if only $\tau' \geqslant \delta > 0$ on (a, b).

This analysis is not fine enough to conclude that a set of positive length has positive parabolic measure at some point (x, t). To obtain that conclusion we apply the integral equation (5.18) three times as before, to obtain successively bounds on g in $L^{3/2}$, L^3 and L^∞, namely,

$$\| g(x, t; y)\|_p \leqslant C \left(1 + \int_a^b |\Gamma(x, t; y, \tau(y))|^3\, dy\right)^{1/3} \tag{5.21}$$

for $p=3/2,\ 3,\ +\infty$. Each inequality is derived by the properties of the operator T mentioned before.

We look more closely at $\Gamma(x, t; y, \tau(y))$ when (x, t) is close to the arc I. As before we let $t-\tau(x)=\lambda>0$, and we shall prove

$$|\Gamma(x, t; y, \tau(y))| \leqslant C\lambda^{-2}. \tag{5.22}$$

Indeed, $|\Gamma(x, t; y, \tau(y))| = O(|t-\tau(y)|^{-3/2})$, so the bound is correct unless $|t-\tau(y)| \leqslant \lambda^{4/3}$. In this case,

$$|\tau(x)-\tau(y)| \geqslant |t-\tau(x)| - \lambda^{4/3} = \lambda - \lambda^{4/3}$$

so that $|y-x| \geqslant c\lambda$ for a certain $c>0$. The maximum value of $t^{-1}W(x, t; 0, 0)$ is $O(x^{-3})$ for all real x. When $|t-\tau(y)| \leqslant \lambda^{4/3}$, we have $|\tau(x)-\tau(y)| \leqslant |t-\tau(y)|+\lambda$, whence $|x-y| \leqslant c\lambda$ for a certain $c>0$. Therefore

$$x-y-\tau'(y)(x-y) = O(\lambda)$$

and

$$|\Gamma(x, t; y, \tau(y))| = O(\lambda)\, O(\lambda^{-3}) = O(\lambda^{-2}).$$

The bound (5.22) is the correct one, for each $(y, \tau(y))$ in I. Taking $x=y-r$, $t=\tau(y)+r^2$, we see that for $r \to 0^+$

$$t-\tau(x) = t-\tau(y)+\tau(y)-\tau(x) = r\tau'(y)+o(r),$$

$$t-\tau(y)-(x-y)\,\tau'(y) = r\tau'(y)+o(r),$$

and

$$\Gamma(x, t; y, \tau(y)) \geqslant cr(r^{-2})^{3/2} \geqslant C'(t-\tau(x))^{-2}.$$

It follows from (5.22) and the observation $\Gamma(x, t; y, \tau(y)) < C\lambda^{-3/2}$ when $|t-\tau(y)| \geqslant \lambda$ that

$$\left(\int_a^b |\Gamma(x, t; y, \tau(y))|^3\, dy\right)^{1/3} = o(\lambda^{-2}). \tag{5.23}$$

By (5.23), (5.21) (5.19) and the fact that (5.22) is sharp, we see that for a certain (x, t), $\lambda(y)$ is positive; since Γ is continuous, a lower bound on λ is valid throughout some neighborhood of y, depending on (x, t).

THEOREM 9. *Let τ be of class $C^1(-\infty, \infty)$ and $\tau' \geqslant \delta > 0$ on $[a, b]$. Let Ω_τ be the region $\{(x, t): t > \tau(x)\}$. Then for each $(y, \tau(y))$, $a<y<b$, there is a point (x, t), where parabolic measure admits a strictly positive continuous density in some neighborhood of $(y, \tau(y))$.*

Consequently, the sets on $\{(x, \tau(x)) \colon a < x < b\}$ *of length* 0 *coincide with the sets of parabolic measure* 0.

We have proved the theorem except for the continuity of $\lambda(y)$. This is obtained from (5.18), the continuity of Γ, the boundedness of λ, and dominated convergence.

On the part of the boundary $\{s = \tau(y)\}$ at which $\tau' = 0$, the situation is unclear. The following lemma and its corollaries contain all of the results on this point.

LEMMA 8. *Let* R *be a rectangle* $\{|x-a| \leqslant h,\ |t-b| \leqslant h^2\}$ *and* V *an open set, regular for the Dirichlet problem. For each point* (x, t) *in* V, *the parabolic measure of* $R \cap \partial V$ *at* (x, t) *with respect to* V *is at most* $A\delta^{-1}h$, *where* A *is an absolute constant and* δ *is the parabolic distance from* (x, t) *to* (a, b): $\delta = |x-a| + |t-b|^{1/2}$.

Proof. Taking $A > 9$ we can suppose that $\delta > 9h$. Now $\delta = |x-a| + |t-b|^{1/2}$, so we have either $|x-a| \geqslant \delta/2$, or $|t-b| \geqslant \delta^2/4 > 20h^2$. The function $u(x, t) = W(x-a, t-b+2h^2)$ is a nonnegative supertemperature, and exceeds ch^{-1} on R and therefore on $R \cap \partial V$. Now either $|x-a| \geqslant \delta/2$, or $|t-b+2h^2| \geqslant |t-b| - 2h^2 \geqslant |t-b|/11 \geqslant \delta^2/50$. Then $u(x, t) \leqslant c'\delta^{-1}$, and the bound for parabolic measure follows from the upper and lower bounds for u.

COROLLARY 1. *Let* Ω_τ *be the region* $\{t > \tau(x)\}$, *where* $|\tau'| \leqslant B$. *Let* (x, t) *have parabolic distance* δ *from* $\partial\Omega_\tau$. *Then an arc* $\{(y, \tau(y)) \colon |y-a| \leqslant h\}$ *has parabolic measure, at* (x, t), *less than* $A\delta^{-1}h + A\delta^{-1}(Bh)^{1/2}$.

Proof. The arc is contained in a rectangle of width $2h$ and height $2Bh$. If $Bh < h^2$, this estimate is already in the lemma. If $Bh > h^2$, i.e., $B > h$, then $h < B^{1/2}h^{1/2}$, and the estimate is $A\delta^{-1}B^{1/2}h^{1/2}$.

COROLLARY 2. *In Corollary* 1 *we suppose instead that* $|\tau''| \leqslant B$. *Then we have instead the estimate* $A\delta^{-1}(1+B)\,h + A\delta^{-1}\,|\tau'(a)|^{1/2}\,h^{1/2}$, *of the parabolic measure.*

Proof. In this case the arc $\{(y, \tau(y)) \colon |y-a| < h\}$ has vertical extent at most $2\,|\tau'(a)|\,h + Bh^2$, and the estimate is derived by the same method as before.

After these preliminaries we can prove

THEOREM 10. *Suppose that* $\tau(x)$ *is of class* $C^2(-\infty, \infty)$. *Then at any* (x, t) *in* Ω_τ, *the parabolic measure on the curve* $s = \tau(y)$ *is absolutely continuous with respect to arc length.*

Proof. On the open set $(\tau' \neq 0)$ we have a stronger property. On the set $Z = \{\tau' = 0\}$, we see by Corollary 2 that the parabolic measure is absolutely continuous, with locally bounded density. At every accumulation point of Z, we have $\tau'' = 0$; since countable sets have parabolic measure 0, a closer examination of the bound derived in Corollary 2 shows that the density is bounded by $A\delta^{-1}$; the upper bound of τ'' does not appear.

Question. We do not know whether a parabolic measure has bounded density even when τ is C^∞ and has compact support, nor whether there are sets of positive length and zero parabolic measure contained in the set $\{\tau' = 0\}$.

Appendix

We first give the proof of Theorem A using Lemma A and then prove Lemma A.

LEMMA A. *Suppose $w(x, t)$ is positive parabolic in R, continuous on $\bar{R}$ and vanishes on ∂R. Then there is a unique Borel measure $d\mu$ on $0 < \lambda < \infty$, and a number $p \geqslant 0$ so that*

$$w(x, t) = \int_0^\infty \sinh \lambda x e^{\lambda^2 t}\, d\mu(\lambda) + px$$

in R.

Proof of Theorem A. Because v is positive parabolic in D, for any $c > 0$ there exists an increasing function α_c on $-\infty < y < \infty$, so that

$$v(x, t) = \int_{-\infty}^\infty W(x, t; y, 0)\, d\alpha_c(y) \tag{A.1}$$

for $-\infty < x < \infty$ and $0 < t < c$; see Theorem 3 in [13, p. 136]. By Theorem 6 of [13, p. 69], for $-\infty < a < b < \infty$

$$\frac{\alpha_c(b^+) + \alpha_c(b^-)}{2} - \frac{\alpha_c(a^+) + \alpha_c(a^-)}{2} = \lim_{t \to 0+} \int_a^b v(x, t)\, dx$$

for any $c > 0$. This says that the measures $d\alpha_c$ and $d\alpha_{c'}$ are the same for any $c, c' > 0$. Define $dV(y, 0)$ to be the unique measure $d\alpha_c$. Equation (0.4) follows from (A.1).

Because u is positive, parabolic in $0 < x < \infty$, $-n < t < n$ for any integer $n > 0$, by Theorem 5.2 of [13, p. 143], there exist increasing functions α_n on $(0, \infty)$ and β_n on $[-n, n)$ so that

$$u(x,t)=\int_{0+}^{\infty} W(x,t;y,-n)-W(x,t;-y,-n)\,d\alpha_n(y)$$
$$+\int_{-n}^{t} K(x,t;0,s)\,d\beta_n(s) \tag{A.2}$$

for $0<x<\infty$ and $-n<t<n$. Let $f_n(x,t)$ and $g_n(x,t)$ be the first and the second integral in (A.2), respectively. Let a_n be a point of continuity of β_n in $(-n,-n+1)$. By Theorem 10.2 of [13, p. 79], we have, for $a_n<b\leqslant n-1$,

$$\beta_n(b^-)-\beta_n(-n)=\lim_{x\to 0+}\int_{-n}^{b} g_n(x,s)\,ds$$

and

$$\beta_n(a_n)-\beta_n(-n)=\beta_n(a_n^-)-\beta_n(-n)=\lim_{x\to 0+}\int_{-n}^{a_n} g_n(x,s)\,ds.$$

Hence for $a_n<b\leqslant n-1$,

$$\beta_n(b^-)-\beta_n(a_n)=\lim_{x\to 0+}\int_{a_n}^{b} g_n(x,s)\,ds.$$

Because $f_n(x,t)$ vanishes continuously on $\{(0,t): a_n\leqslant t\leqslant n-1\}$, it follows from the above equality that for $a_n\leqslant b\leqslant n-1$,

$$\beta_n(b^-)-\beta_n(a_n)=\lim_{x\to 0+}\int_{a_n}^{b} u(x,s)\,ds. \tag{A.3}$$

Hence, if $k\geqslant n$,

$$d\beta_n=d\beta_k \qquad \text{on} \quad (-n+1,\, n-1).$$

Define $dU=d\beta_n$ on $(-n+1,\, n-1)$ and let

$$g(x,t)=\int_{-\infty}^{t} K(x,t;0,s)\,dU(s).$$

Fix $0<x<\infty$ and $-\infty<t<\infty$; then

$$u(x,t)\geqslant g_n(x)\geqslant\int_{-n+1}^{t} K(x,t;0,s)\,dU(s)$$

for every integer n satisfying $-n<t$. Hence

$$u(x,t)\geqslant g(x,t).$$

Standard method shows that $f_n(x,t)$ and $g(x,t)-g_n(x,t)$ vanish continuously on $\{x=0\}\times\{-n+1<t<n-1\}$ when (x,t) approaches that part of the boundary. Therefore $u(x,t)-g(x,t)$ vanishes continuously on $\{x=0\}\times\{-n+1<t<n-1\}$ for each $n>0$, hence on $\{x=0\}\times\{-\infty<t<\infty\}$.

Since $u(x,t)-g(x,t)$ is positive, parabolic in R, vanishes continuously on ∂R, the representation (0.3) follows from Lemma A.

The uniqueness of p, dU and $d\mu$ follows from (A.3) and Lemma A.

Proof of Lemma A. Given any integer $n>0$, from Corollary 5.2b of [13, p. 146] and the fact that $\lim_{(x,t)\to(0,-n)} w(x,t)=0$, it follows that there exists an increasing function ρ_n on $0<y<\infty$, so that

$$w(x,t)=\int_{0+}^{\infty} W(x,t;y,-n)-W(x,t;-y,-n)\,d\rho_n(y) \qquad \text{(A.4)}$$

for $0<x<\infty$ and $-n<t<1$.

We denote $W(x,t;y,-n)-W(x,t;-y,-n)$ by $W(x,t;y,-n)$ and rewrite (6.4) as

$$\begin{aligned} w(x,t)&=\int_{0+}^{\infty}\frac{\tilde W(x,t;y,n)}{\tilde W(1,0;y,n)}\,\tilde W(1,0;y,n)\,d\rho_n(y)\\ &=\int_{0+}^{\infty}\frac{\tilde W(x,t;\lambda n,n)}{\tilde W(1,0;\lambda n,n)}\,\tilde W(1,0;\lambda n,n)\,d\alpha_n(\lambda), \end{aligned} \qquad \text{(A.5)}$$

where α_n is an increasing function on $0<\lambda<\infty$.

For fixed $x>0$, $t<0$ and $M>0$, one may verify that

$$\frac{\tilde W(x,t;\lambda n,n)}{\tilde W(1,0;\lambda n,n)} \qquad \text{(A.6)}$$

converges uniformly to $\sinh(\lambda x/2)\,e^{\lambda^2 t/4}(\sinh(\lambda/2))^{-1}$ on the interval $0<\lambda<M$ as $n\to\infty$.

For fixed $n>0$, $x>0$, $-n/2<t<0$, we claim that

$$\frac{\tilde W(x,t;\lambda n,n)}{\tilde W(1,0;\lambda n,n)}\leqslant C(1+x)\,e^{-\lambda^2|t|/4+\lambda x+1/(4n)}, \qquad \text{(A.7)}$$

where C is a positive absolute constant. We recall that

$$\begin{aligned} W(x,t;y,n)&=\frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{t+n}}\left(e^{-(x-y)^2/(4(t+n))}-e^{-(x+y)^2/(4(t+n))}\right)\\ &=\frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{t+n}}\,e^{-(x-y)^2/(4(t+n))}\left[1-e^{-xy/(t+n)}\right]. \end{aligned}$$

For fixed $n > 0$, $x > 0$ and $-n/2 < t < 0$,

$$\begin{aligned}\frac{(y-x)^2}{n+t} - \frac{(y-1)^2}{n} &> \left(\frac{y^2}{n+t} - \frac{2xy}{n+t}\right) - \left(\frac{y^2}{n} + \frac{1}{n}\right) \\ &= \frac{y^2\,|t|}{n(n+t)} - \frac{2xy}{n+t} - \frac{1}{n},\end{aligned} \tag{A.9}$$

which attains its minimum value

$$\frac{-nx^2}{|t|\,(n+t)} - \frac{1}{n} = O(1) \tag{A.10}$$

at $y = nx/|t|$. Let $y = \lambda n$, we obtain from (A.9)

$$\begin{aligned}\frac{(y-x)^2}{(n+t)} - \frac{(y-1)^2}{n} &> \frac{n^2\lambda^2\,|t|}{n(n+t)} - \frac{2xn\lambda}{n+t} - \frac{1}{n} \\ &> \lambda^2|t| - 4\lambda x - \frac{1}{n}.\end{aligned} \tag{A.11}$$

It is easy to verify that

$$1 - e^{-\lambda nx/(t+n)} \leqslant \min\{1, 2x\lambda\} \tag{A.12}$$

and

$$\begin{aligned}1 - e^{-\lambda n/(t+n)} &\geqslant 1/2 \qquad \text{if} \quad \lambda \geqslant 1 \\ &\geqslant \lambda/2 \qquad \text{if} \quad 0 < \lambda < 1.\end{aligned} \tag{A.13}$$

Combining (A.8), (A.9), (A.10), (A.11), (A.12) and (A.13) we conclude that, for fixed $n > 0$, $x > 0$, $-n/2 < t < 0$, (A.7) holds, and

$$\tilde{W}(x, t; \lambda n, n)/\tilde{W}(1, 0; \lambda n, n) = O(1) \tag{A.14}$$

as $\lambda \to \infty$.

Because

$$\int_{0+}^{\infty} \tilde{W}(1, 0; \lambda n, n)\, d\alpha_n(\lambda) = w(1, 0) < \infty, \tag{A.15}$$

a subsequence of $\tilde{K}(1, 0; \lambda n, n)\, d\alpha_n(\lambda)$ converges weakly to a measure $d\beta$ on $0 \leqslant \lambda < \infty$. Fix $M > 0$, $n > 0$, $x > 0$ and $-n/2 < t < 0$, it follows from (A.7) and (A.15) that

$$\int_{M}^{\infty} \tilde{W}(x, t; \lambda n, n)\, d\alpha_n(\lambda) \leqslant Cw(1, 0)(1 + x)\, e^{-M^2|t|/4 + Mx + 1/(4n)}. \tag{A.16}$$

By (A.6) and (A.14)

$$\lim_{n\to\infty}{}' \int_{0+}^{M} \frac{\tilde{W}(x, t; \lambda n, n)}{\tilde{W}(1, 0; \lambda n, n)} \tilde{W}(1, 0; \lambda n, n)\, d\alpha_n(\lambda) = \int_0^M f(x, t)\, d\beta(\lambda), \quad (A.17)$$

where by $\lim'_{n\to\infty}$, we mean a limit along an appropriate subsequence, and

$$\begin{aligned} f(x, t) &= \sinh\frac{\lambda x}{2} e^{\lambda^2 t/4} \left(\sinh\frac{\lambda}{2}\right)^{-1}, \qquad 0 < \lambda < M \\ &= \lim_{\lambda\to 0} \sinh\frac{\lambda x}{2} e^{\lambda^2 t/4} \left(\sinh\frac{\lambda}{2}\right)^{-1} = x, \qquad \lambda = 0. \end{aligned}$$

Because (A.16) and (A.17) hold for arbitrarily large M and n, in view of (A.5) we conclude that for $x > 0$ and $t < 0$,

$$w(x, t) = \int_{0^+}^{\infty} \sinh\frac{\lambda x}{2} e^{\lambda^2 t/4} \left(\sinh\frac{\lambda}{2}\right)^{-1} d\beta(\lambda) + \beta(0^+)\, x. \qquad (A.18)$$

In fact there exist increasing functions μ_n on $0 < \lambda < \infty$ and numbers $p_n \geqslant 0$ so that for $0 < x < \infty$, $-\infty < t < n$,

$$w(x, t) = \int_{0+}^{\infty} \sinh \lambda x e^{\lambda^2 t}\, d\mu_n(\lambda) + p_n x. \qquad (A.19)$$

A change of variable of (A.18) gives (A.19) in the case $n = 0$. For $n > 0$ the proof is similar to that of $n = 0$.

We observe that $p_n = \lim_{t\to-\infty} W(1, t) \equiv p$.

We claim that the measure $d\mu_n$ in (A.19) is unique and is independent of n. Suppose

$$\begin{aligned} \int_{0+}^{\infty} \sinh \lambda x e^{\lambda^2 t}\, d\mu_n(\lambda) &= \int_{0+}^{\infty} \sinh \lambda x e^{\lambda^2 t}\, d\mu_m(\lambda) \\ &= w(x, t) - px \qquad (A.20) \end{aligned}$$

for $0 < x < \infty$ and $-\infty < t < \min\{m, n\}$. Fix such a t and let

$$d\nu_m(\lambda) = e^{\lambda^2 t}\, d\mu_m(\lambda)$$

and

$$d\nu_n(\lambda) = e^{\lambda^2 t}\, d\mu_n(\lambda).$$

and rewrite (A.20) as

$$\int_{0+}^{\infty} \sinh \lambda x\, d\nu_m(\lambda) = \int_{0+}^{\infty} \sinh \lambda x\, d\nu_n(\lambda) < +\infty. \qquad (A.21)$$

We observe that

$$\int_{0+}^{\infty} \sinh \lambda z \, dv_m(\lambda) \equiv \int_{0+}^{\infty} \sinh \lambda z \, dv_n(\lambda) \tag{A.22}$$

for all complex z. Because of (A.21), both sides of (A.22) are entire functions of z; and again because of (A.21), they must be identical. Moreover, we have

$$\int_{0+}^{\infty} \lambda \cosh \lambda z \, dv_m(\lambda) = \int_{0+}^{\infty} \lambda \cosh \lambda z \, dv_n(\lambda). \tag{A.23}$$

By letting $z = it$ and $dv_m(-\lambda) = -dv_m(\lambda)$ and $dv_n(-\lambda) = -dv_n(\lambda)$ for $\lambda > 0$, we conclude from (A.23) that

$$\int_{-\infty}^{\infty} e^{i\lambda t} \lambda \, dv_n(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} \lambda \, dv_n(\lambda).$$

Hence $dv_m = dv_n$ on $0 < \lambda < \infty$ and

$$d\mu_m = d\mu_n \qquad \text{on} \quad 0 < \lambda < \infty.$$

Define $d\mu = d\mu_n$ for any n and conclude Lemma A.

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